

Spin Foam Models of Matter Coupled to Gravity

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Abstract

We construct a class of spin foam models describing matter coupled to gravity, such that the gravitational sector is described by the unitary irreducible representations of the appropriate symmetry group, while the matter sector is described by the finite-dimensional irreducible representations of that group. The corresponding spin foam amplitudes in the four-dimensional gravity case are expressed in terms of the spin network amplitudes for pentagrams with additional external and internal matter edges. We also give a quantum field theory formulation of the model, where the matter degrees of freedom are described by spin network fields carrying the indices from the appropriate group representation. In the non-topological Lorentzian gravity case, we argue that the matter representations should be appropriate $SO(3)$ or $SO(2)$ representations contained in a given Lorentz matter representation, depending on whether one wants to describe a massive or a massless matter field. The corresponding spin network amplitudes are given as multiple integrals of propagators which are matrix spherical functions.

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1 Introduction

The Barret-Crane spin foam state sum models of four-dimensional quantum gravity [1, 2, 3, 4], represent a promising approach for constructing a quantum theory of gravity. These are non-topological state sum models, and they give discrete spacetime three-geometry to three-geometry transition amplitudes [5]. One can consider these models as examples of regularized path integral approach to quantum gravity. The spacetime is described by a simplicial complex, and the state sum is over colored triangulations. This approach is a category theory generalization of the Regge calculus, where the basic category involved is the category of representations of the appropriate symmetry group. This group is taken to be $SO(d)$ in the d -dimensional Euclidian gravity case, or $SO(d-1, 1)$ in the Lorentzian case. Other groups, like anti de-Sitter, can be considered as well [5].

One can also think of the spin foam models as the higher-dimensional generalizations of the string theory matrix models, since the spin foam amplitudes are given by a field theory Feynman diagrams which are dual to spacetime triangulations [8, 9, 10, 5]. In the $d = 2$ case one recovers the string theory matrix models [5, 11].

The standard way of making the spin foam partition function and the amplitudes finite is by passing to the corresponding quantum group category [6, 7, 2]. However, it was discovered in [3, 4] that even in the Lie group case, the BC model could be made finite by changing the field theory propagator. It has been proven that the BC Euclidian model is perturbatively finite [12], and recently this has been proven for the BC Lorentzian model [13], i.e. the amplitude for any non-degenerate finite triangulation of the spacetime is finite.

In order to make a spin foam model a serious candidate for a quantum theory of gravity, like string theory, one should be able to incorporate matter. Mathematically this would mean enlarging the category of representations, by adding a new set of representations corresponding to the matter fields. The first proposal of this kind was made by Crane [14], who suggested a tensor category of unitary representations for the quantum Lorentz group. In this proposal a subset of the irreducible representations (irreps) corresponds to the gravitational degrees of freedom (DOF), while the rest corresponds to the matter DOF. The corresponding state sum model is topological, which is justified as an acceptable feature of the theory at high energies. Then one conjectures that the local dynamics would appear at low energies, by some

sort of symmetry breaking mechanism. Although an interesting proposal, it is not clear how to understand the matter DOF, which are represented by infinite-dimensional unitary Lorentz irreps. It is not known how to relate these DOF to the particle field theory description in terms of fields carrying pseudo unitary finite-dimensional Lorentz irreps.

In this paper we give a formulation for a class of spin foam models describing matter coupled to gravity, where the gravitational sector is described by the unitary irreps of the appropriate symmetry group, while the matter sector is described by the finite dimensional irreps of that group. In the non-topological four-dimensional Lorentzian gravity case, the gravitational sector will be described by the BC model, while in the matter sector we will argue that one needs to take appropriate $SO(3)$ or $SO(2)$ representations contained in a given finite-dimensional Lorentz irrep. The construction we present here naturally follows from the field theory formulation of the spin foam amplitudes [5]. It was shown there that a spin foam transition amplitude can be represented as an matrix element of an evolution operator, where the initial and the final states correspond to the three-dimensional spin networks (or spin nets for short) induced by the boundary triangulations. One can represent these spin net states as the spin net operators acting on an appropriate Fock space vacuum. One can think of these spin net operators as the second quantized spin net wave-functions/states from the canonical loop gravity [15]. Since the spin net wave-functions/states carrying fermionic matter have been introduced in the context of canonical loop gravity [16, 17], it is not difficult to generalize these results to the second quantization formalism[†], so that the matter insertions at the spin net vertices become appropriate creation/annihilation operators.

In section 2 we give basic definitions and formulas concerning the spin net functions. In section 3 we give basics on spin foams, and explain how the spin foam amplitudes are related to the spin net amplitudes. In section 4 we construct the fermionic spin net operators and states, and describe the general structure of the interaction with the gravitational background. In section 5 we discuss the fermions in the Euclidian gravity case. We show that in the non-topological gravity case, the corresponding spin-net amplitudes can be written as the multiple integrals of propagators over the homogenous

[†]The spin network quantum field theory is an example of what is traditionally called a third quantization of gravity. Since general relativity is a classical field theory, then the expression “second quantization” is a more appropriate terminology.

space $S^3 = SO(4)/SO(3)$, where the fermionic propagator is given by a matrix zonal spherical function. These results are straightforwardly generalized to the Lorentzian gravity case in the next section. In section 7 we discuss the general structure of the interaction terms for the case of matter fields of arbitrary spins. In the non-topological gravity case, the spin-net matter propagators are described by the corresponding matrix spherical functions. We also argue that the spin-net propagators for the analogues of the massless gauge bosons are the matrix spherical functions restricted to an $SO(2)$ subgroup. In section 8 we present our conclusions.

2 Spin net functions and amplitudes

We review the results on spin nets [18, 19] from a point of view which is most suitable for the type of generalizations we want to make.

Consider square-integrable functions Ψ on G^E , where G is a Lie group and E is a natural number. A gauge invariant function is determined by an oriented graph Γ with V vertices and E edges, where a group element is associated to each edge $e \in E$. If we label the vertices of Γ as v_j , $1 \leq j \leq V$, then the edges connecting the vertices j and k can be denoted as e_{jk} , and the corresponding group elements as g_{jk} . Orientation is such that $g_{jk} = g_{kj}^{-1}$. A function Ψ_Γ associated to the graph Γ is gauge invariant if

$$\Psi_\Gamma(g_{jk}) = \Psi_\Gamma(h_j g_{jk} h_k^{-1}) \quad , \quad (1)$$

for all $h_j, h_k \in G$.

By using the Peter-Weyl theorem, one can show that a gauge invariant function Ψ_Γ can be expanded as

$$\Psi_\Gamma(g_1, \dots, g_E) = \sum_{\Lambda, i} c_{\Lambda, i} \Phi_{\Gamma, \Lambda, i}(g_1, \dots, g_E) \quad (2)$$

where $\Lambda = \{\Lambda_1, \dots, \Lambda_E\}$ denote the unitary irreps of G labeling the edges of Γ , and $i = \{i_1, \dots, i_V\}$ denote the intertwiners associated to the vertices of Γ . The basis functions Φ are called the spin net functions, and they are given by

$$\prod_{e \in E} D^{(\Lambda)}_{\alpha_e}{}^{\beta_e}(g_e) \prod_{v \in V} C_{\dots}^{(i_v)} \quad , \quad (3)$$

where $D^{(\Lambda)}(g)$ is the representation matrix of g , acting in the space $V(\Lambda)$. All the representation indices α_e and β_e are contracted by the indices from the

product of the intertwiner components. Contraction is defined as $X^\alpha Y_\alpha = C^{\alpha\beta} X_\alpha Y_\beta$, where $C^{\alpha\beta}$ are the components of the $V(\Lambda) \rightarrow V(\Lambda^*) = V^*(\Lambda)$ intertwiner, and V^* is the dual vector space. Since any intertwiner can be represented as a linear map $V(\Lambda_1) \otimes \cdots \otimes V(\Lambda_n) \rightarrow \mathbf{C}$, we denote the corresponding vector space as $Inv(\Lambda_1, \dots, \Lambda_n)$.

Given a spin net function $\Phi_{\Gamma, \Lambda, i}(g_e)$, one can obtain a group invariant associated to the spin net

$$A(\Gamma, \Lambda, i) = \Phi_{\Gamma, \Lambda, i}(g_1 = g_0, \dots, g_E = g_0) \quad , \quad (4)$$

where g_0 is the identity element of G . We will call the number A the amplitude for the spin net (Γ, Λ, i) . These objects are the basic building blocks of the state sum amplitudes. Since $D_\alpha^\beta(g_0) = \delta_\alpha^\beta$, the amplitude A is simply the product of the intertwiners $C_{\dots}^{\dots(i_v)}$ for all vertices $v \in V$, with all the representation indices being contracted. The graphs Γ which are relevant for the state sum models are the one skeletons of d -simplices, as well as the Θ_d graphs (two points joined by d edges).

In the case of open spin nets, the above results can be easily generalized. The amplitude A for an open spin net will be a group tensor given by the product of intertwiners for the vertices, where all the representations indices are contracted, except the indices corresponding to the irreps labelling the external edges. This can be represented as

$$A_{\alpha_1 \dots}^{\beta_2 \dots}(\Gamma, \Lambda, i) = C_{\dots}^{\dots(i_1)} \dots C_{\dots \alpha_1 \dots}^{\dots(i_k)} \dots C_{\dots}^{\dots \beta_2 \dots(i_l)} \dots C_{\dots}^{\dots(i_V)} \quad . \quad (5)$$

Hence the amplitude for an open spin net is not an invariant, but it is a tensorial quantity. One can construct a group invariant by contracting the external edge indices with the vectors and the co-vectors from the external irrep vector spaces

$$S(\Gamma, \Lambda, i, v, u) = A_{\alpha_1 \dots}^{\beta_2 \dots}(\Gamma, \Lambda, i) v_{\beta_2} \dots u^{\alpha_1} \dots \quad , \quad (6)$$

where v_α are the components of a vector $v \in V(\Lambda)$, while u^α are the components of a covector $u \in V^*(\Lambda)$. This formula will be useful for constructing the invariant actions, since it is the group theory description of the actions used in particle field theories.

It is now also clear how to construct the open spin net functions $\Phi_{\alpha_1 \dots}^{\beta_2 \dots}(g_e)$. Note that the expansions of the functions over G^d used in the field theory reformulations of the d -dimensional spin foam amplitudes as Feynman graphs

[8, 9, 10, 5], are simply expansions in terms of open spin net functions associated to the \times_d graphs (d edges emanating from a single vertex).

In the case of the BC model [1, 2], the amplitude for the \times graph is not just the intertwiner $C_{\times}^{(i)} \in \text{Inv}(\Lambda_1, \dots, \Lambda_4)$, but it is a linear combination of these

$$S_{\times} = \sum_i \alpha_i C_{\times}^{(i)} \quad . \quad (7)$$

This is the consequence of the fact that the subcategory of irreps which are used have invariant vectors under the $SO(3)$ subgroup.

3 Spin foam amplitudes

The state sum models which have been used for constructing d -dimensional quantum gravity models are based on a two-complex $J = (V, E, F)$, consisting of a finite number of vertices $v \in V$, edges $e \in E$ and faces $f \in F$. The two complex J is taken to be dual to a triangulation of the spacetime manifold M . Since the faces of J are labeled with the irreps of G , one can regard this object as a generalization of the spin net, where the one-complex $\Gamma = (V, E)$ is replaced by a set (E, F) from the two complex J . That is why J is called a spin foam [18]. Hence these state sum models give the amplitudes for the spin foams. An amplitude for a spin foam can be written as

$$A(J) = \sum_{\Lambda} \prod_{f \in F} A_2(\Lambda_f) \prod_{e \in E} A_1(\Lambda_{f_1(e)} \cdots \Lambda_{f_k(e)}) \prod_{v \in V} A_0(\Lambda_{f_1(v)} \cdots \Lambda_{f_l(v)}) \quad , \quad (8)$$

where A_2 is the amplitude for a face, A_1 is the amplitude for an edge, and A_0 is the amplitude for a vertex of the two complex J .

There are several ways to determine the amplitudes A_n [18], but what is important for our purposes is the fact that these amplitudes are the amplitudes for the closed spin nets defined in the previous section. In the $d = 4$ spacetime case, A_0 is the amplitude for the 4-simplex graph (a pentagram) [9, 1, 2], while A_1 can be the amplitude for four parallel lines, or it is the Θ_4 graph in the modified BC model [3, 4, 13]. A_2 is always $\dim \Lambda$, which is again a single loop spin net amplitude.

Another aspect of the spin foam amplitude (8) which will be important is the fact that it can be represented as a Feynman diagram of a field theory over the group G [8, 9, 10, 5]. The action for this field theory can be

written in terms of the Fourier modes ϕ , which transform like tensors from $Inv(\Lambda_1, \dots, \Lambda_d)$ spaces. In the $d = 4$ case the action takes the form

$$\begin{aligned} S_4 &= \frac{1}{2} \sum_{\Lambda, i} Q_{\vec{\alpha}_1 \vec{\alpha}_2}(\times_1, \times_2) \phi_{\times_1}^{\vec{\alpha}_1} \phi_{\times_2}^{\vec{\alpha}_2} \\ &+ \frac{1}{5!} \sum_{\Lambda, i} \mathcal{V}_{\vec{\alpha}_1 \dots \vec{\alpha}_5}(\times_1, \dots, \times_5) \phi_{\times_1}^{\vec{\alpha}_1} \dots \phi_{\times_5}^{\vec{\alpha}_5} \quad , \end{aligned} \quad (9)$$

where $\phi_{\times}^{\vec{\alpha}} = \phi_{\Lambda_1 \dots \Lambda_4(i)}^{\alpha_1 \dots \alpha_4}$. The vertex functions Q and \mathcal{V} are given by

$$Q^{-1}(\times_1, \times_2) = \delta^4 A_1(\Lambda_1, \dots, \Lambda_4; i_1, i_2) \quad (10)$$

$$\mathcal{V}(\times_1, \dots, \times_5) = \delta^{10} A_0(\Lambda_1, \dots, \Lambda_{10}; i_1, \dots, i_5) \quad , \quad (11)$$

where δ are the appropriate delta functions of the indices. A_1 can be the amplitude for the Θ_4 graph, or it could be one, when is the amplitude for four parallel lines. A_0 is the amplitude for the pentagram. In the case of BC models, A_0 and A_1 will not depend on the intertwiner labels i_k .

The corresponding Feynman diagram is just the one complex (V, E) obtained from the two complex $J = (V, E, F)$. This diagram is a $(d+1)$ -valent graph in the case of d -dimensional model. Its value is given by (8), which can be represented by a collection of spin net diagrams, such that to each vertex we associate a d -simplex graph, while to each edge we associate d parallel lines, or a Θ_d graph in the modified version.

4 Fermionic matter

The Fourier modes $\phi_{\times}^{\vec{\alpha}}$ can be promoted into creation and annihilation operators [5], so that the spin foam amplitude can be represented as an matrix element of an evolution operator, in analogy with the particle field theory case. As shown in [5], one can construct the “in” and the “out” states which describe the spatial spin nets corresponding to the boundary triangulations of the spacetime manifold. Such spin net states can be represented by

$$|\gamma\rangle = \prod_{v \in V(\gamma)} \phi_{\times_v} |0\rangle \quad , \quad (12)$$

where $|0\rangle$ is the vacuum of the Fock space constructed from the creation and annihilation operators ϕ and ϕ^\dagger , γ is a four-valent spin net graph dual to

the boundary triangulation, and all the representation indices α_v are appropriately contracted. We take ϕ^\dagger to be an annihilation operator, in order to make the notation simpler.

In the first quantization formalism [16, 17], the fermionic matter can be introduced by replacing the spin net γ with an open spin net γ_F , which is obtained from γ by putting an external edge carrying a fermionic irrep Λ_s at each site of γ where a fermion is located. One can also put more than one fermion at each lattice site, up to $\dim \Lambda_s$ (Pauli exclusion principle). In that case, the external edge is labeled with an irrep from an antisymmetrized tensor product of k Λ_s irreps [17].

In the second quantization formalism [5], we will construct the $|\gamma_F\rangle$ state by introducing the fermionic creation and annihilation operators $\psi_\sigma(\times_v)$ and $\psi_\sigma^\dagger(\times_v)$, where σ is the representation index of Λ_s , and the label \times_v denotes the four irreps of the spin net site where the fermion is located. Hence the corresponding state would be given as

$$|\gamma_F\rangle = \prod_{v' \in V'(\gamma)} \prod_{\sigma_{v'}} \psi_{\sigma_{v'}}(\times_{v'}) \prod_{v \in V(\gamma)} \phi_{\times_v} |0\rangle, \quad (13)$$

where V' is the set of vertices where the fermions are located.

In order to calculate the transition amplitudes between two fermionic spin net states, which would be a fermionic spin foam amplitude, we need to introduce the corresponding terms in the field theory action (9). Since classically the fermionic propagation is described by a line in the spacetime, and since the fermions interact with the gravitational field, the candidate action will be

$$S_F = \sum_{\Lambda} \psi^\sigma(\times) \psi^{\sigma'}(\times') \mathcal{V}_{\sigma\sigma'}(\times, \times', \times_1, \dots, \times_n) \phi_{\times_1} \cdots \phi_{\times_n} + (h.c.) \quad , \quad (14)$$

where $(h.c.)$ stands for the hermitian conjugate term.

The vertex \mathcal{V} can be determined from two requirements. First, S_F has to be a group invariant, i.e. a scalar. Second, we require that the fermions propagate only on complexes which are dual to the Feynman graphs of the gravitational model (9). The idea behind this is that the purely gravitational model builds up the space-time on which the fermions propagate. Hence $n = 5$, and

$$\mathcal{V}_{\sigma\sigma'}(\times, \times', \times_1, \dots, \times_5) = \delta^{10} \delta_{\times, \times_1} \delta_{\times', \times_k} A_{\sigma\sigma'}(\Lambda_1, \dots, \Lambda_{10}), \quad (15)$$

where $A_{\sigma\sigma'}$ is a spin net amplitude for the pentagram with two external edges attached at site 1 and site k , where $1 \leq k \leq 5$.

We will also add to S_F a purely quadratic fermion term

$$\sum_{\Lambda} \psi^{\sigma}(\times) Q_{\sigma\sigma'}(\times, \times') \psi^{\sigma'}(\times') + (h.c.) \quad , \quad (16)$$

in order to have a well-defined perturbative expansion. The propagator Q^{-1} will be determined by the amplitude for the Θ_4 graph with two external edges labelled with Λ_s , or it could be a trivial amplitude for five parallel lines. We denote the propagator amplitude as $G_{\sigma\sigma'}$.

Note that $(\psi_{\sigma})^* = \psi^{\sigma} = C^{\sigma\sigma'} \psi_{\sigma'}$, and the vector space duality $*$ is in general different from the complex conjugation (reality) properties. The reality properties determine relation between the creation and annihilation operators ψ and ψ^{\dagger} [5].

Therefore the maximal closed spin net subgraph for $A_{\sigma\sigma'}$ has to be at least the pentagram graph. Hence a relevant fermionic Feynman diagram Γ_F which is generated by the $S_4 + S_F$ action will be determined by a spacetime skeleton diagram Γ (a 5-valent graph dual to a spacetime triangulation) plus a line connecting the vertices of Γ . Therefore Γ_F will be a graph consisting of 5-valent and 7-valent vertices, see Fig. 1. Out of these diagrams we will consider only those for which the fermion path is a line which connects the centers of a string of adjacent 4-simplices, extending from the initial to the final boundary.

Hence the fermionic spin foam J_F will be given by the usual spin foam J and a line L of edges starting from an “in” edge and terminating with an “out” edge. The corresponding amplitude will be given by

$$A_{\sigma_i}^{\sigma_f}(J_F) = \sum_{\Lambda} \prod_{f \in F} A_f(\Lambda) A_{\sigma_i}^{\sigma_f}(L) \prod_{e \in E'} A_e(\Lambda) \prod_{v \in V'} A_v(\Lambda), \quad (17)$$

where

$$A_{\sigma_i}^{\sigma_f}(L) = A_{\sigma_i}^{\sigma_1}(v_1) G_{\sigma_1}^{\sigma_2}(e_{12}) A_{\sigma_2}^{\sigma_3}(v_2) \cdots G_{\sigma_{k-1}}^{\sigma_k}(e_{k-1,k}) A_{\sigma_k}^{\sigma_f}(v_k) \quad . \quad (18)$$

The number of “in” and “out” fermions has to be the same for a free theory, and the above formula can be then easily generalized to the case when there are several lines L .

The amplitude A for a fermionic vertex of J_F will be given by a spin net amplitude for a pentagram with two external fermionic edges, Fig. 2

$$A_{\sigma'}^{\sigma}(\Lambda_1, \dots, \Lambda_{10}) = \sum_{\alpha} C_{\sigma' \times} C_{\times}^{\sigma} C_{\times} C_{\times} C_{\times} \quad , \quad (19)$$

where $C_{\sigma' \times}, C_{\times}^{\sigma} \in \text{Inv}(\Lambda_s, \Lambda_1, \dots, \Lambda_4)$ and $C_{\times} \in \text{Inv}(\Lambda_1, \dots, \Lambda_4)$. The amplitude for an fermionic edge of J_F will be given by the propagator amplitude $G_{\sigma}^{\sigma'}$, which in the non-trivial case is given by the amplitude for the Θ_4 graph with two external edges, Fig. 3

$$G_{\sigma'}^{\sigma}(\Lambda_1, \dots, \Lambda_4) = \sum_{\alpha} C_{\sigma' \times} C_{\times}^{\sigma} \quad . \quad (20)$$

In some of the cases the minimal amplitudes (19) and (20) are zero, because $C_{\sigma \times} = 0$. Then one can replace it with $C_{\sigma \sigma' \times}$ from $\text{Inv}(\Lambda_s, \Lambda_s, \Lambda_1, \dots, \Lambda_4)$, so that

$$A_{\sigma'}^{\sigma}(\Lambda_1, \dots, \Lambda_{10}) = \sum_{\alpha} C_{\sigma' \times}^{\tau} C_{\tau \times}^{\sigma} C_{\times} C_{\times} C_{\times} \quad . \quad (21)$$

The corresponding spin net graph is a pentagram with two external and one internal fermion edge, joined in a line, Fig. 4. Similarly, for the edge amplitude $G_{\sigma}^{\sigma'}$, we will have in the non-trivial case the Θ_5 graph with two external edges, Fig. 5, so that

$$G_{\sigma'}^{\sigma}(\Lambda_1, \dots, \Lambda_4) = \sum_{\alpha} C_{\sigma' \tau \times} C_{\times}^{\tau \sigma} \quad . \quad (22)$$

5 Euclidian case

We now study the four-dimensional Euclidian gravity spin foam model [1, 3], so that $G = SO(4)$. The irreps of $SO(4)$ can be labeled as $\Lambda = (j, k)$ where j and k are the half-integers labelling the irreps of $SU(2)$. In the topological gravity case, one uses all $SO(4)$ irreps for labelling the triangles of a triangulation of M . The fermions will be represented by $\Lambda_s^+ = (\frac{1}{2}, 0)$ and $\Lambda_s^- = (0, \frac{1}{2})$ irreps. These irreps correspond to chiral, or Weyl fermions, while a Dirac fermion would be a reducible representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.

The relevant intertwiners for constructing the fermionic spin foam amplitude are from the space $\text{Inv}(\Lambda_s, \Lambda_1, \dots, \Lambda_4)$. A basis $|i\rangle$ is determined by

$$C_{\sigma \alpha_1 \dots \alpha_4}^{\Lambda_s \Lambda_1 \dots \Lambda_4} (i) = \sum_{\alpha, \alpha'} C_{\sigma \alpha_1 \alpha}^{\Lambda_s \Lambda_1 \Lambda} C_{\alpha \alpha_2 \alpha'}^{\Lambda \Lambda_2 \Lambda'} C_{\alpha' \alpha_3 \alpha_4}^{\Lambda' \Lambda_3 \Lambda_4} \quad . \quad (23)$$

In order to simplify the notation, we take $X_{\alpha} Y_{\alpha} = X^{\alpha} Y_{\alpha} = C^{\alpha \beta} X_{\alpha} Y_{\beta}$. Note that the formula (23) can be represented graphically by Fig. 6. Hence the intertwiner label is given by $i = (\Lambda, \Lambda')$, and we can now construct the fermionic amplitudes for topological gravity spin foam model by using

the procedure described in sect. 4. However, we will not discuss further the topological gravity case, and we will concentrate on the non-topological model, because of its importance for physics.

The set of relevant irreps in the non-topological gravity case is given by the simple irreps $N = (j, j)$. In that case $C_{\dots\dots}^{\Lambda_s N_1 \dots N_4} = 0$, which can be proven by using a Z_2 grading of the $SO(4)$ irreps. This grading is introduced by splitting the irreps into even and odd, or bosonic and fermionic irreps via the parity function $p(n) = \{-1, 1\}$ if n is odd, even, respectively, as

$$p(j, k) = p(2j)p(2k) \quad . \quad (24)$$

Hence the simple irreps are even, i.e. bosonic, while Λ_s^\pm are odd, i.e. fermionic. Since $p(\Lambda_1 \otimes \Lambda_2) = p(\Lambda_1)p(\Lambda_2) = p(\Lambda)$ where Λ is an irrep from the tensor product, then a product of bosonic irreps can never yield a fermionic irrep, and hence $C^{\Lambda_s N_1 \dots N_4} = 0$. Therefore the minimal option for the intertwiner space is $Inv(\Lambda_s, \Lambda_s, N_1, \dots, N_4)$, and a basis $|i\rangle$ can be constructed from

$$C_{\sigma\sigma'\alpha_1\dots\alpha_4}^{\Lambda_s\Lambda_s N_1\dots N_4(i)} = \sum_{\beta_1, \beta_2, \beta_3} C_{\sigma\sigma'\beta_1}^{\Lambda_s\Lambda_s B_1} C_{\alpha_1\alpha_2\beta_2}^{N_1 N_2 B_2} C_{\beta_1\beta_2\beta_3}^{B_1 B_2 B_3} C_{\beta_3\alpha_3\alpha_4}^{B_3 N_3 N_4} \quad (25)$$

where B are the bosonic irreps. The equation (25) can be graphically represented by Fig. 7.

Now we have to impose the constraints coming from the fact that the simple irrep vector spaces have invariant vectors under the $SO(3) \cong SU(2)$ subgroup. This amounts to constructing an invariant vector in the corresponding $Inv(n)$ space. When there are no fermions, then $n = 4$, and the invariant vector, or the BC vertex, is constructed as [1]

$$S_{\alpha_1\dots\alpha_4}^{N_1\dots N_4} = \sum_M C_{\alpha_1\dots\alpha_4}^{N_1\dots N_4(M)} = \sum_M \sum_\alpha C_{\alpha_1\alpha_2\alpha}^{N_1 N_2 M} C_{\alpha\alpha_3\alpha_4}^{M N_3 N_4}, \quad (26)$$

where M are the simple irreps. The $3j$ symbols for $SO(4)$ can be expressed as the product of $3j$ symbols for $SU(2)$ as

$$C_{\alpha_1\alpha_2\alpha_3}^{N_1 N_2 N_3} = C_{m_1 m_2 m_3}^{j_1 j_2 j_3} C_{n_1 n_2 n_3}^{j_1 j_2 j_3} \quad . \quad (27)$$

The consequence of using the BC vertex for constructing the spin net amplitudes is that these amplitudes can be represented as multiple integrals over the coset space $X = G/H$, which is a three-sphere $S^3 = SO(4)/SO(3)$ [20, 21]. For example, the amplitude for the Θ_4 graph will be given by

$$\Theta(N_1, \dots, N_4) = \sum_\alpha S_{\alpha_1\dots\alpha_4}^{N_1\dots N_4} S_{\alpha_1\dots\alpha_4}^{N_1\dots N_4} \quad . \quad (28)$$

One can show that

$$\Theta(N_1, \dots, N_4) = \int_{X^2} dx dy \prod_{i=1}^4 K_{N_i}(x, y) \quad , \quad (29)$$

where

$$K_N(x, y) = \text{Tr } D^{(j)}(x^{-1} \cdot y) = \frac{\sin((2j+1)\theta)}{\sin \theta} \quad , \quad (30)$$

and θ is the geodesic distance between the coset points. This formula works because the coset points x and y can be considered as $SU(2)$ group elements due to equivalence of S^3 and $SU(2)$ spaces.

We will now use this property to construct the fermionic BC vertex. Consider the following integral associated to the Θ_4 graph

$$\Theta_{ss'}(N_1, \dots, N_4) = \int_{X^2} dx dy K_{ss'}(x, y) \prod_{i=1}^4 K_{N_i}(x, y) \quad , \quad (31)$$

where

$$K_{ss'}(x, y) = D_{ss'}^{(1/2)}(x^{-1} \cdot y) \quad . \quad (32)$$

One can now show that

$$\Theta_{ss'}(N_1, \dots, N_4) = \sum_{\alpha, \bar{s}} S_{\alpha_1 \dots \alpha_4 \bar{s} s}^{N_1 \dots N_4 N_{1/2}} S_{\alpha_1 \dots \alpha_4 \bar{s} s'}^{N_1 \dots N_4 N_{1/2}} \quad , \quad (33)$$

where S is an element of $\text{Inv}(N_1, \dots, N_4, N_{1/2})$ given by

$$S_{\alpha_1 \dots \alpha_4 \sigma}^{N_1 \dots N_4 N_{1/2}} = \sum_{M, N} \sum_{\alpha, \beta} C_{\alpha_1 \alpha_2 \alpha}^{N_1 N_2 M} C_{\alpha \sigma \beta}^{M N_{1/2} N} C_{\beta \alpha_3 \alpha_4}^{N N_3 N_4} \quad , \quad (34)$$

and $N_{1/2} = (\frac{1}{2}, \frac{1}{2})$.

When (34) is compared to (25), then it is easy to see that the expression (34) can be also taken as an intertwiner from the space $\text{Inv}(\Lambda_s^+, \Lambda_s^-, N_1, \dots, N_4)$, due to $N_{1/2} = \Lambda_s^+ \otimes \Lambda_s^-$. Therefore the expression (31) can be interpreted as the amplitude for the open spin net consisting of the Θ_4 graph plus two external edges labelled with Λ_s^+ and one internal edge labeled with Λ_s^- , Fig. 5.

Hence the action S_F for the Λ_s fermions will be given by

$$\begin{aligned} S_F &= \sum_N \psi_{\pm}^{\sigma}(\times) Q_{\sigma\nu}(\times, \times') \psi_{\pm}^{\nu}(\times') \\ &+ \sum_N \psi_{\pm}^{\sigma}(\times) \psi_{\pm}^{\nu}(\times') \mathcal{V}_{\sigma\nu}(\times, \times'; \times_1 \dots \times_5) \phi_{\times_1} \dots \phi_{\times_5} \\ &+ (h.c.) \quad , \end{aligned} \quad (35)$$

where Q^{-1} and \mathcal{V} are determined by the

$$G_{\sigma\nu}(N_1\dots N_4) = \int_{X^2} d^2x K_{\sigma\nu}(x_1, x_2) \prod_i K_{N_i}(x_1, x_2) \quad (36)$$

and

$$A_{\sigma\nu}(N_1\dots N_{10}) = \int_{X^5} d^5x K_{\sigma\nu}(x_1, x_k) \prod_{i < j} K_{N(ij)}(x_i, x_j) \quad (37)$$

spin net amplitudes, respectively. The edge amplitude (36) is determined by the graph from Fig. 5, and the vertex amplitude (37) is determined by the graph from Fig. 4.

The reason why (31) and (33) are the same is that the spinor propagator (32) is an example of a matrix spherical function [22], which is a generalization of the scalar propagator K_N (30). In general case [22], if H is a subgroup of G and if τ is an irrep of H , consider irreps Λ of G which contain τ when decomposed with respect to the subgroup H , i.e.

$$V_\Lambda = \bigoplus_{1 \leq k \leq n} V_\tau^k \oplus \dots = H_\tau \oplus \dots \quad (38)$$

Let $\Phi(g) = P_\tau D^{(\Lambda)}(g) P_\tau$, where P_τ is the projector from V_Λ to H_τ . $\Phi(g)$ are called τ -spherical functions, and $P_\tau^k D^{(\Lambda)}(g)$ are eigenfunctions of the Laplacian on the homogenous vector bundle $E_\tau = (G/H, V_\tau)$. Since $\Phi(g) \in \text{Hom}(H_\tau, H_\tau)$, we can construct $\phi(g) = \text{tr} \Phi(g) \in \text{Hom}(V_\tau, V_\tau)$, where tr is a partial trace over the degeneracy label k . Then consider the object

$$K_{\Lambda, \tau}(x, y) = U(x, x_0) \phi(g_x^{-1} g_y) U(x_0, y) \quad , \quad (39)$$

where U 's are vector bundle parallel transport operators along the geodesics (x_0, x) and (x_0, y) , and $g_x = \sigma(x)$ is a local section of the principal bundle $(G/H, G)$. By using $g_x = xh$, $g_y = y\tilde{h}$, $U(x, x_0) = D^{(\tau)}(h)$ and $U(x_0, y) = D^{(\tau)}(\tilde{h}^{-1})$, one can show that K is a function of only the coset space points. Note that when τ is the identity irrep, the expression (39) reduces to the scalar propagator K_N , where N is the class one irrep. Hence (39) is the matrix generalization of K_N , and we can use it to construct the spin net amplitudes for graphs with external edges.

In the $SO(4)$ case, let $\vec{\mathcal{A}}$ and $\vec{\mathcal{B}}$ be the generators of the two commuting $SU(2)$. Then the $SO(4)$ rotations generators $J_{\mu\nu}$ can be expressed as

$$\vec{J} = \vec{\mathcal{A}} + \vec{\mathcal{B}} \quad , \quad \vec{K} = \vec{\mathcal{A}} - \vec{\mathcal{B}} \quad , \quad (40)$$

where $J_i = \epsilon_i^{jk} J_{jk}$ and $K_i = J_{i0}$. The $SO(4)$ irreps Λ are then determined by the pairs of half-integers (A, B) , where A and B are the angular momenta numbers of the two $SU(2)$. We then have

$$\begin{aligned} D^{(A,B)}(\vec{\theta}, \vec{v}) &= \exp(i\vec{\theta}\vec{J}_\Lambda + i\vec{v}\vec{K}_\Lambda) \\ &= \exp(i\vec{\theta}_+\vec{\mathcal{A}}_A + i\vec{\theta}_-\vec{\mathcal{B}}_B) \\ &= D^{(A)}(\vec{\theta}_+) \otimes D^{(B)}(\vec{\theta}_-), \end{aligned} \quad (41)$$

where

$$\vec{\theta}_\pm = \vec{\theta} \pm \vec{v} \quad . \quad (42)$$

The diagonal $SU(2)$ subgroup vectors are given by

$$|j, m\rangle = C_{mab}^{jAB} |a\rangle \otimes |b\rangle \quad . \quad (43)$$

These vectors form an irrep space of the spatial rotation generators \vec{J} . In the case of Λ_s , we take $\tau = j = 1/2$, so that $P_\tau = 1$. From (41) we get

$$K_{1/2}^E(x, y) = D^{(1/2)}(x^{-1} \cdot y) = D^{(1/2)}(-\vec{v}) D^{(1/2)}(\vec{u}) \quad , \quad (44)$$

where

$$D^{(1/2)}(\vec{r}) = \cos(r/2)I_2 + i\vec{\sigma}\vec{n} \sin(r/2) \quad , \quad \vec{n} = \frac{\vec{r}}{\sqrt{(\vec{r})^2}} = \frac{\vec{r}}{r} \quad , \quad (45)$$

and $\vec{\sigma}$ are the Pauli matrices.

6 Lorentzian case

In the Lorentzian case $G = SO(3, 1) \cong SL(2, C)$, and the finite-dimensional irreps (j, k) become pseudo unitary, because the relation (40) becomes

$$\vec{J} = \vec{\mathcal{A}} + \vec{\mathcal{B}} \quad , \quad \vec{K} = (-i)(\vec{\mathcal{A}} - \vec{\mathcal{B}}) \quad , \quad (46)$$

so that the boost generator \vec{K} becomes an anti-hermitian operator. This is the reason why there are no finite-dimensional unitary irreps for the Lorentz group.

Hence the unitary Lorentz irreps are infinite-dimensional, and can be labelled as (m, ρ) , where m is a half-intiger, while ρ is a non-negative real

number [23]. The gravitational DOF are again described by the simple irreps, or the class one irreps with respect to the spatial $SU(2)$ subgroup. These are given by the $(m, 0)$ and $(0, \rho)$ irreps [2]. However, only the $(0, \rho)$ irreps are used [2]. We could then use the pseudo unitary irreps (j, k) to describe the matter fields, but we have to check first the consistency of the state-sum model with unitary and pseudo unitary irreps of the Lorentz group.

We now have $N = (0, \rho)$ in the gravitational sector. The corresponding propagator is given by [2]

$$K_\rho(x, y) = \frac{\sin(\rho d(x, y))}{\rho \sinh d(x, y)} \quad , \quad (47)$$

where $d(x, y)$ is the geodesic distance between the points x and y of the coset space X , which is now a hyperboloid $SO(1, 3)/SO(3)$. The fermions will carry the irreps Λ_s^\pm . The tensor product decomposition rule of a unitary and a pseudo-unitary irrep can be inferred from the decomposition of these irreps under the $SU(2)$ subgroup. For the (m, ρ) irrep we have [24]

$$\mathcal{H}_{(m, \rho)} = \bigoplus_{j=|m|}^{\infty} V_j \quad . \quad (48)$$

Since $V(\Lambda_s^\pm) = V_{\frac{1}{2}}$, we obtain

$$\mathcal{H}_{(m, \rho)} \otimes V(\Lambda_s^\pm) = \mathcal{H}_{(m-\frac{1}{2}, \rho)} \oplus \mathcal{H}_{(m+\frac{1}{2}, \rho)} \quad , \quad (49)$$

which agrees with [14]. This implies

$$\mathcal{H}_{(m, \rho)} \otimes V(\Lambda_v) = \mathcal{H}_{(m-1, \rho)} \oplus \mathcal{H}_{(m, \rho)} \oplus \mathcal{H}_{(m, \rho)} \oplus \mathcal{H}_{(m+1, \rho)} \quad , \quad (50)$$

where $\Lambda_v = (\frac{1}{2}, \frac{1}{2})$.

Hence $C^{\rho_1 \rho_2 \Lambda_v} \neq 0$, and we can now construct the Lorentzian analog of the BC spinor vertex (34). For this we need a Lorentzian spinor propagator, which is determined by the formula (39). It is given by an analytic continuation of the Euclidian formula (44), where $\vec{v} \rightarrow -i\vec{v}$, so that

$$K_{1/2}^L(x, y) = D^{(1/2)}(x^{-1} \cdot y) = D^{(1/2)}(i\vec{v}) D^{(1/2)}(-i\vec{u}) \quad , \quad (51)$$

where now

$$D^{(1/2)}(i\vec{r}) = \cosh(r/2) I_2 + \vec{\sigma} \vec{n} \sinh(r/2) \quad . \quad (52)$$

7 Interactions and gauge fields

Note that in the discussion in section 4 one could have taken Λ_s to be an arbitrary irrep of G , and moreover, one can take a set of different irreps $\Lambda_{s_1} = S_1, \dots, \Lambda_{s_k} = S_k$ to represent fields of different spins. In the topological gravity case, one can write the action as

$$\begin{aligned}
S_I &= \sum_S \sum_{\Lambda} \psi_S(\times) \psi_S(\times') Q_{SS}(\times, \times') \\
&+ \sum_S \sum_{\Lambda} \psi_S(\times) \psi_S(\times') \mathcal{V}_{SS}(\times, \times', \times_1, \dots, \times_5) \phi_{\times_1} \cdots \phi_{\times_5} \\
&+ \sum_S \sum_{\Lambda} \psi_{S_1}(\times) \cdots \psi_{S_k}(\times^{(k)}) \mathcal{V}_{S_1 \dots S_k}(\times, \dots, \times^{(k)}; \times_1, \dots, \times_5) \phi_{\times_1} \cdots \phi_{\times_5} \\
&+ (h.c.) \quad , \tag{53}
\end{aligned}$$

where the vertex Q will be determined by the spin net amplitudes for the open graphs based on the five parallel lines and the Θ_4 , while the vertex \mathcal{V} will be determined by the spin net amplitudes for open graphs based on the pentagram. A new feature is that the set of possible graphs for the vertex $\mathcal{V}_{S_1 \dots S_k}$ will contain graphs with more than five vertices, Fig. 8.

In the non-topological gravity case, we need to implement the H invariance. This requires a specification of the irreps s of H which are contained in the irreps S of G . In that case the matter fields will have the indices from the H irreps, and the action can be written as

$$\begin{aligned}
S_I &= \sum_S \sum_N \psi_s(\times) \psi_s(\times') Q_{ss}^S(\times, \times') \\
&+ \sum_S \sum_N \psi_s(\times) \psi_s(\times') \mathcal{V}_{ss}^S(\times, \times', \times_1, \dots, \times_5) \phi_{\times_1} \cdots \phi_{\times_5} \\
&+ \sum_S \sum_N \psi_{s_1}(\times) \cdots \psi_{s_k}(\times^{(k)}) \mathcal{V}_{s_1 \dots s_k}^{S_1 \dots S_k}(\times, \dots, \times^{(k)}; \times_1, \dots, \times_5) \phi_{\times_1} \cdots \phi_{\times_5} \\
&+ (h.c.) \quad , \tag{54}
\end{aligned}$$

where the vertices Q and \mathcal{V} will be determined by the spin net amplitudes for open graphs based on five parallel lines, the Θ_4 and the pentagram graphs, as in the topological case. However, these amplitudes will be now given as multiple integrals of the propagators $K_N(x, y)$ and $K_{S,s}(x, y)$.

In the Euclidian/Lorentzian gravity case, it is instructive to study the case of spin $j = 1$ matter fields, because in nature the fermions interact through the spin $j = 1$ gauge bosons. If we take a vector irrep $S_1 = (\frac{1}{2}, \frac{1}{2})$,

it will contain a $s = j = 1$ irrep of $SO(3)$. We denote the corresponding spin net field as $A_i(\times)$, while the fermions we denote as $\psi_\sigma^\pm(\times) = \psi_s^\pm(\times)$, where the subscripts \pm denote the corresponding $SO(4)$ irreps. The spin foam analog of the particle field theory interaction term $\int d^4x \bar{\psi} \gamma^\mu \psi A_\mu$ can be written as

$$\sum_N \psi_s^+(\times) \psi_{s'}^-(\times') A_i(\times'') \mathcal{V}_{+-}^{s s' i}(\times, \times', \times''; \times_1, \dots, \times_5) \phi_{\times_1} \cdots \phi_{\times_5} + (h.c.), \quad (55)$$

where the vertex \mathcal{V}_{+-} is determined by the spin net amplitude for the open graph given by the pentagram plus a three-valent matter vertex, Fig. 9. This amplitude is given by

$$A_{+-}^{s s' i} = \int_{X^6} d^5x dy K_{\frac{1}{2}}^{st}(x_1, y) K_{(\frac{1}{2}, \frac{1}{2}), 1}^{ij}(x_4, y) K_{\frac{1}{2}}^{s't'}(x_2, y) C_{t t' j}^{\frac{1}{2} \frac{1}{2} 1} \prod_{k < l} K_{N_{kl}}(x_k, x_l). \quad (56)$$

Since (55) is a spin foam generalization of the particle gauge field theory $U(1)$ interaction, it is natural to ask what is the analog of the $U(1)$ gauge symmetry. One way to deal with the issue of gauge symmetry is to assume that the components of the fields appearing in the action are the physical, or the independent DOF. Note that one can adopt this type of approach in the context of flat space particle field theory [25]. One simply takes the creation and annihilation operators labelled by the irreducible Poincare representation indices and then forms polynomial interaction terms via the formula (6). The relevant Poincare irreps are the massive and the massless ones, so that the index labels are the three-momentum, the $SO(3)$ irrep indices for the massive case, or the two-dimensional $SO(2) \cong Spin(2)$ representation labels in the massless case.

Since one wants the particle interactions to be local in spacetime, one then constructs fields carrying appropriate Lorentz irreps from the creation and annihilation (c/a) operators, and writes the interaction terms as spacetime integrals over the polynomials of these fields [25]. The problem with this approach is that the massless fields cannot be always expressed in terms of the helicity c/a operators. This happens for the vector field $A_\mu(x)$, which when expressed in terms of c/a operators transforms non-homogenously under the Lorentz transformations [25]. However, the field $f_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ transforms like a Lorentz tensor, so that the cubic interaction term would be given by $\int d^4x \bar{\psi} \gamma^{\mu\nu} \psi f_{\mu\nu}$. But this is not the type of interaction realized in nature, which is $\int d^4x \bar{\psi} \gamma_\mu \psi A^\mu$. This paradox is resolved by introducing the $U(1)$ gauge transformations [25].

In the case of our spin foam models, this type of problem will not appear immediately, because one is dealing only with the c/a operators for the Lorentz group and its subgroups. These operators carry the labels $\times = (N_1 \dots N_4)$ and finite-dimensional representation indices. There are no Poincare irreps c/a operators, and the reason for this is that the particle momentum is not conserved in curved spacetimes. Hence it is not clear what would be a physical interpretation of a spin-foam model based on the Poincare group.

This can be seen on the example of a Poincare spin foam model based on the simple unitary irreps of the Lorentz group. At first sight this appears a natural thing to do, because the unitary irreps of the Lorentz group are also unitary irreps of the Poincare group, since the Lorentz subgroup is the little group for the zero momentum. Hence the gravitational sector would be described by the zero-momentum unitary irreps, while the matter sector would be described by the usual particle unitary irreps, having a non-zero momentum. However, because the tensor product conserves the momentum, the corresponding spin-foam amplitude would conserve the momentum of a particle propagating in a curved background.

Hence we take the components of our fields to be the physical or the independent local canonical coordinates. This makes sense since our matter operators live on the spin nets defined by the triangulations of three-space boundaries, see (13). We can then consider these operators as the discretized analogs of the reduced phase space canonical formalism fields. The canonical analysis of gravity plus matter actions shows that the effect of the local Lorentz and the spacetime diffeomorphism constraints is to reduce the number of components of the gravitational field, while the matter field components are affected only by the gauge symmetry constraints. The final result is that the non-gauge fields have independent components which transform as the appropriate representations of $SO(3)$, while the gauge fields independent components transform as the appropriate representations of $SO(2)$. In the flat space limit, these independent component fields become Fourier transforms of the Poincare irreps, so that the non-gauge fields correspond to the massive irreps, while the gauge fields correspond to the massless ones[‡]. This fact indicates that there is a hidden local Poincare invariance. This is plausible, since the local Lorentz invariance plus the spacetime diffeomorphism invariance can be seen as the local Poincare invariance. However, this aspect

[‡]Except in the case of massless spin-half fermions, or if there is a Higgs scalar field.

is not well understood.

Therefore the $A_i(\times)$ operator cannot be a discrete massless gauge field analog, since it carries a $j = 1$ irrep of $SO(3)$. In order to introduce a massless-like vector gauge boson in the model, we need a spin net field which transforms as a two-dimensional representation of $SO(2)$. Therefore we need to construct invariant propagators under an $SO(2)$ subgroup of the $SO(3)$ subgroup.

Hence consider a subgroup H' of H , and let τ' be a representation (could be reducible) of H' contained in the irrep τ of H . Note that one cannot use the $K_{\Lambda, \tau'}$ propagators for this purpose, where τ' is contained in the irrep Λ . The reason is that $K_{\Lambda, \tau'}$ will be a function on the coset space $X' = G/H'$, which is different from the space $X = G/H$, where the gravitational propagators K_N are defined. Therefore these propagators can not be multiplied and integrated to give the spin net amplitudes.

Note that the representation τ' will induce a vector sub-bundle $(G/H, V_{\tau'})$ of the vector bundle $(G/H, V_{\tau})$. We can then define the propagator

$$\tilde{K}_{\Lambda, \tau'}(x, y) = P_{\tau'} U(x, x_0) \phi(g_x^{-1} g_y) U(x_0, y) P_{\tau'} \quad (57)$$

$$= P_{\tau'} K_{\Lambda, \tau}(x, y) P_{\tau'} \quad , \quad (58)$$

where $P_{\tau'}$ is a projector from V_{τ} to $V_{\tau'}$. It is clear that $\tilde{K} \in Hom(V', V')$ and it transforms correctly under $x \rightarrow xh'$ and $y \rightarrow y\tilde{h}'$, where $h', \tilde{h}' \in H'$.

In our case we take the index on the massless-like spin net field of spin j to be from a reducible $Spin(2)$ helicity representation

$$V'_j = \mathcal{L}\{|j\rangle, |-j\rangle\} \quad . \quad (59)$$

This choice is motivated by the continuum space canonical analysis and with the flat space Poincare invariance. Clearly V'_j is a subspace of V_j , and the $Spin(2)$ subgroup is generated by the J_z component of the angular momentum.

In the $j = \frac{1}{2}$ case we have $V(\Lambda_s^{\pm}) = V_{\frac{1}{2}} = V'_{\frac{1}{2}}$, so that the only way to distinguish between the massive and a massless fermion is by a choice of the terms in the quadratic action (35).

In the $j = 1$ and $S_1 = (\frac{1}{2}, \frac{1}{2})$ case, the index i in the massless-like A_i operator will take two values, so that the cubic interaction term would be again given by the formula (56), but now all the indices will belong to the $SO(2)$ $V'_{\frac{1}{2}}$ and V'_1 representations.

Note that in the $j = 1$ case one can also take $S_1^+ = (1, 0)$ and $S_1^- = (0, 1)$ Lorentz irreps. These are the irreducible parts of the antisymmetric tensor $f_{\mu\nu}$, and we can denote the corresponding spin net fields as $f_i^\pm(\times)$. The spin-foam analog of the action $\int d^4x \bar{\psi} \gamma^{\mu\nu} \psi f_{\mu\nu}$ will be given by

$$\sum_N \psi_s^\pm(\times) \psi_{s'}^\pm(\times') f_i^\pm(\times'') \mathcal{V}_{\pm\pm}^{s s' i}(\times, \times', \times''; \times_1, \dots, \times_5) \phi_{\times_1} \cdots \phi_{\times_5} + (h.c.), \quad (60)$$

and the corresponding spin net amplitude is given by

$$A_{\pm\pm}^{s s' i} = \int_{X^6} d^5x dy K_{\frac{1}{2}}^{st}(x_1, y) K_1^{ij}(x_4, y) K_{\frac{1}{2}}^{s't'}(x_2, y) C_{tt'j}^{\frac{1}{2} \frac{1}{2} 1} \prod_{k < l} K_{N_{kl}}(x_k, x_l), \quad (61)$$

where $K_1 = K_{(1,0),1} = K_{(0,1),1}$.

Note that the construction (59) gives for $j = 2$ a massless-like helicity two field $h_{\pm 2}(\times)$. Since this is a local excitation on a given spin net, it can be considered as a spin foam version of the graviton. Again there are two possible choices for the corresponding Lorentz irrep. One can take $S_2 = (1, 1)$ or $S_2^+ = (2, 0)$ and $S_2^- = (0, 2)$. The first choice corresponds to a symmetric traceless tensor $h_{\mu\nu}$, while the second and the third choice are the irreducible pieces of the tensor $R_{\mu\nu, \rho\sigma}$, which has the symmetry properties of the Riemann curvature tensor. Clearly, we will then take the $(1, 1)$ irrep for the graviton, and one would then use the $\tilde{K}_{(1,1),2}$ propagator to construct the interaction vertices.

The realistic interactions require an introduction of the internal symmetry group \tilde{G} . In the framework of the spin foam models, the simplest way to do this is to replace the category $Cat(G)$ with the category $Cat(G \times \tilde{G})$. The matter irreps would be then given by (S, λ_S) , where λ_S is the corresponding irrep of \tilde{G} . The gravitational sector irreps would be $(N, I_{\tilde{G}})$. The matter action would again have the form (53) in the topological gravity case, or (54) in the non-topological gravity case, but now the products of the matter fields have to be contracted by the intertwiners $C_{\dots\dots}^{\lambda_1 \dots \lambda_k}$ for the internal group irreps.

8 Conclusions

The fact that we have taken the matter spin net fields in the non-topological gravity model to have indices from representations of $SO(3)$ and $SO(2)$ subgroups, means that the transition amplitudes will be $SO(3)$ or $SO(2)$ group

covariant objects. One then wanders what happens with the Lorentz covariance. There is no such problem for the spin-half fermions, or for the $(j, 0)$ and the $(0, j)$ matter irreps. However, if we want to have the most general interactions, we need to include (j, k) irreps with $jk \neq 0$. Hence we encounter a spin-foam analogue of the particle field theory spinor-vector interaction problem. The only possible answer is that there should be a spin-foam version of the gauge invariance which reduces the Lorentz covariant expressions to $SO(3)$ or $SO(2)$ covariant gauge-fixed expressions. In that case the expressions (54) and (55) can be understood as gauge-fixed actions. One can think of these actions as a spin-foam generalization of the light-cone gauge field theory actions [26]. Actually, this analogy with the light-cone gauge field theory is even closer, since the finite-dimensional $SO(n)$ irreps are unitary, and hence the corresponding spin-foam model involves only the unitary representations, in contrast to the topological gravity spin foam model, where the matter irreps are pseudo-unitary.

Obtaining Lorentz covariant expressions in the non-topological gravity case would then amount to coupling simple unitary irreps N to pseudo-unitary irreps S , and then forming the actions of the type (53). These actions should be invariant under a spin-foam version of the gauge transformations, so that one would obtain the action (54) in a particular gauge. A better understanding of these issues deserves a further study.

Note that one can try to resolve the problem of internal symmetry group gauge invariance by using the Wilson loop operators instead of the connections. It would be interesting to study this approach in the context of gravitational spin foam models. However, it is not obvious how to couple the Wilson loop irreps (I_G, λ) to the gravitational irreps $(N, I_{\tilde{G}})$.

An attractive feature of our formalism is that one can introduce a spin-foam analogue of the graviton. This means that there are two types of gravitational excitations in our formalism. The ϕ_\times operators create the spatial spin networks, i.e. the three-space and its geometry, while the field theory Feynman diagrams give the amplitudes for the spin foams, which describe evolution from a three-geometry to a three-geometry via the corresponding colored triangulation of the spacetime. The $h(\times)$ operators could then describe fluctuations of a given background three-geometry, and the corresponding spin-foam amplitudes will then describe the propagation of these fluctuations in the spacetime, i.e. the propagation of gravitational waves. Hence it is important to study further the model, in order to better understand the continuous spacetime limit, since there is a possibility that one

would recover the perturbative gravity in a fixed background spacetime geometry.

An important issue which has to be explored is the question of the perturbative finiteness of the model. As in the purely gravitational case [12, 13], this will boil down to the study of the asymptotic behaviour of the spin-net propagators K_N and $K_{S,s}$. Note that in the Euclidian case $Tr K_{S,s}$ is the same as K_N , where $N = (s, s)$, so that one expects that the Euclidian spin-foam matter model should be finite. In the Lorentzian case there is no such a simple relation between the propagators, so that one needs to make a more detailed analysis. Alternatively, one can formulate a quantum group version of the construction we gave, which could be made automatically finite if the gravitational irreps are from a finite set, and if there are finitely many matter irreps.

The issue of the semiclassical/continuous space limit of the spin foam models needs a more detailed study. There are no rigorous statements about the semiclassical theory. In the purely gravitational case one can show that in the limit of large angular momenta one obtains amplitudes which are sums of terms proportional to $\exp(\pm iS_R)$, where S_R is the Regge form of the Einstein-Hilbert action [27]. Actually, because of the form of the propagator K_N , see (30) and (47), this seems to be a good approximation even for arbitrary values of the angular momenta. Hence there are good indications that one is on the right track. In the case of matter, one would then need to obtain expressions containing $\exp(\pm iS_R \pm iS_{sm})$, where S_{sm} is the simplicial version of the matter actions [28]. This type of analysis could also settle the question of which fields are really massive and which ones are massless.

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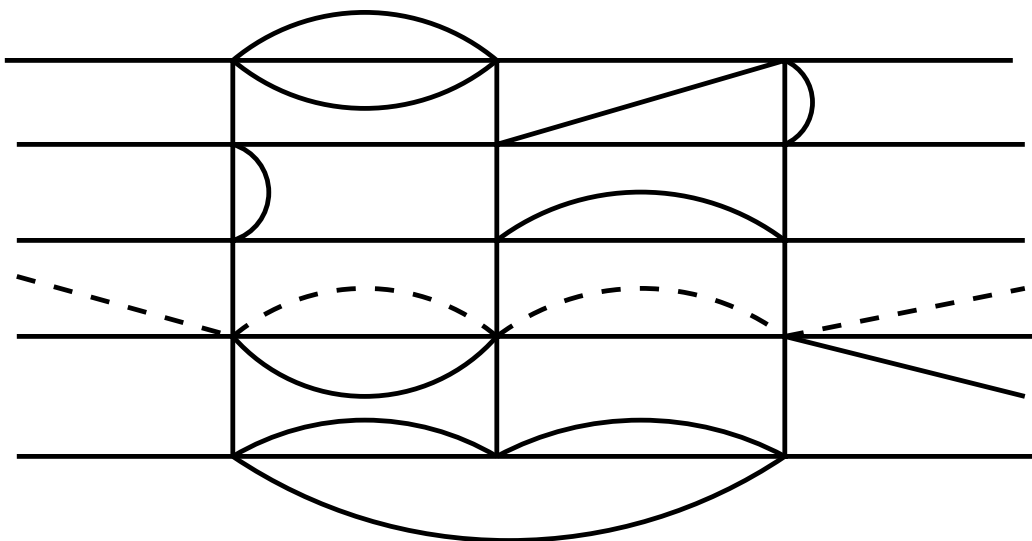


Figure I: A spin foam Feynman diagram describing propagation of a matter field, which is represented by a dotted line.

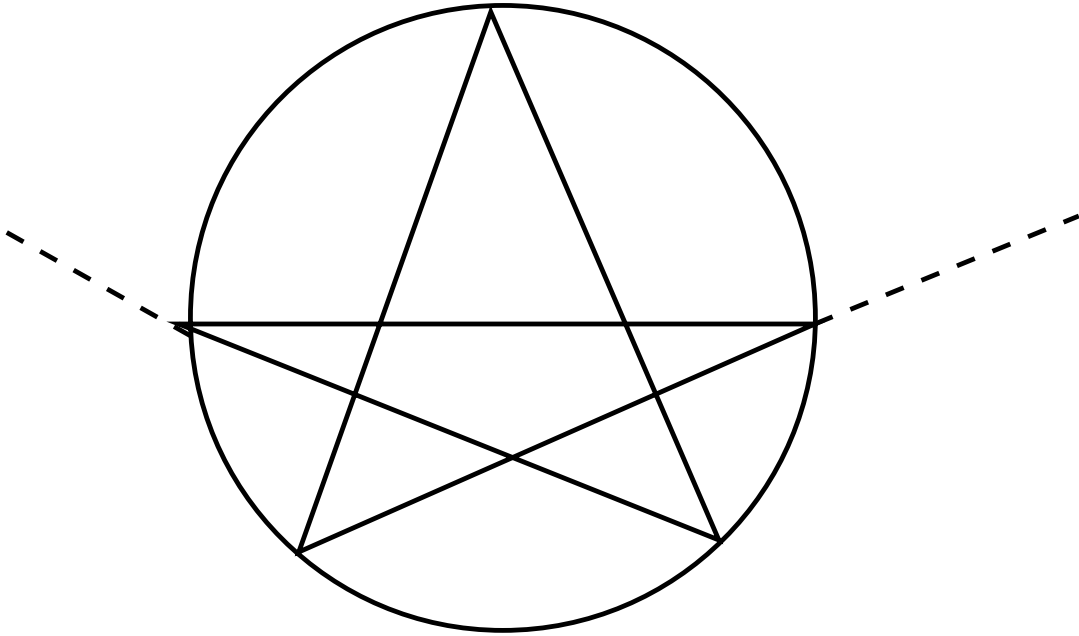


Figure II: The simplest spin net graph which gives a 7-valent vertex for a spin foam Feynman diagram.

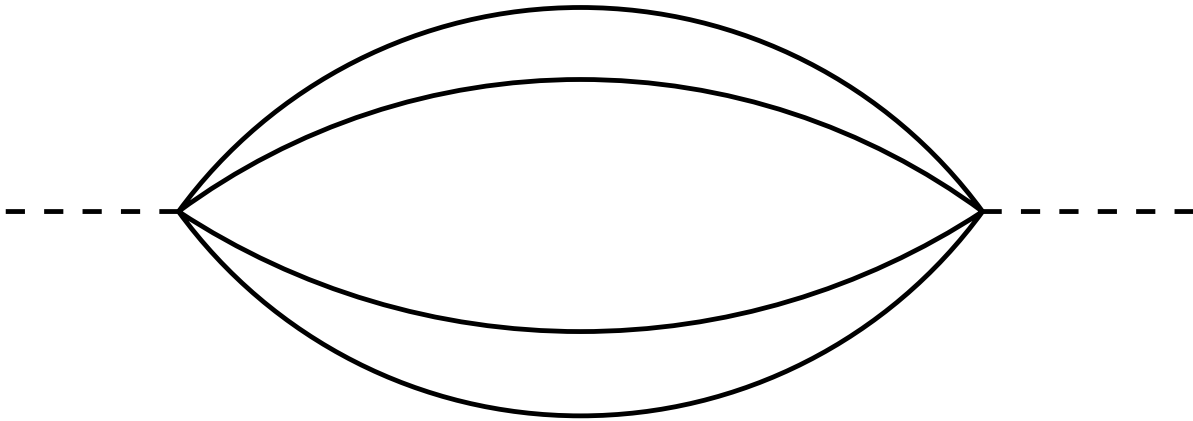


Figure III: A nontrivial spin net graph which gives a matter propagator for a spin foam Feynman diagram.

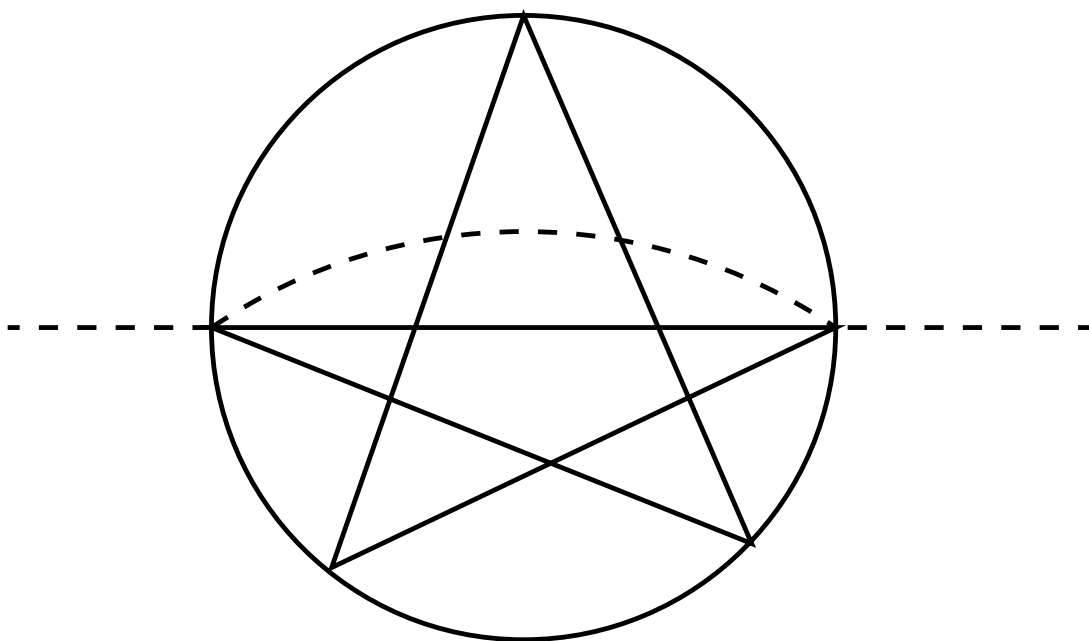


Figure IV: An example of a more complex spin net graph determining the 7-valent spin foam vertex.

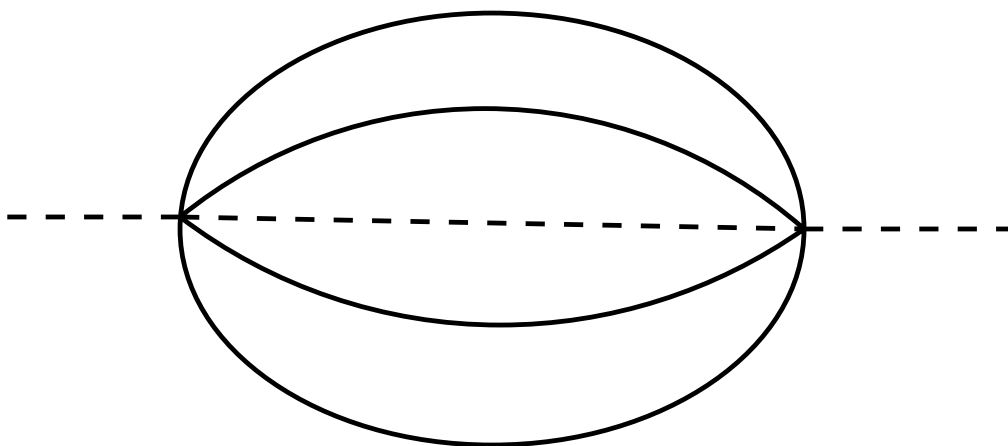


Figure V: A more complex spin net graph determining the matter spin foam propagator.

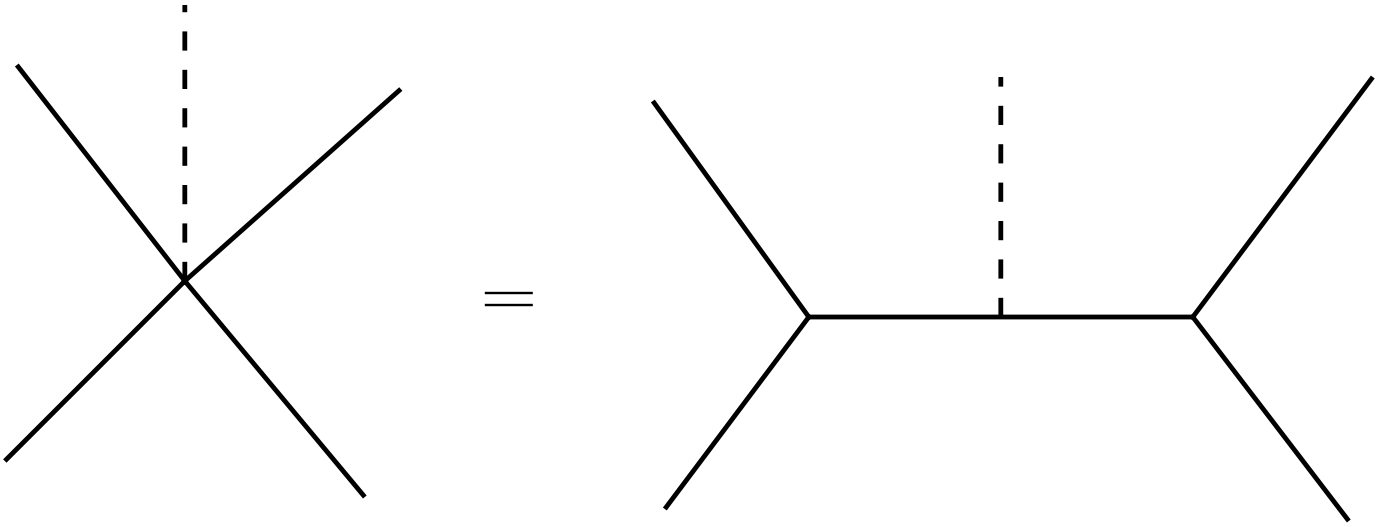


Figure VI: Diagrammatic representation of the equation (23).

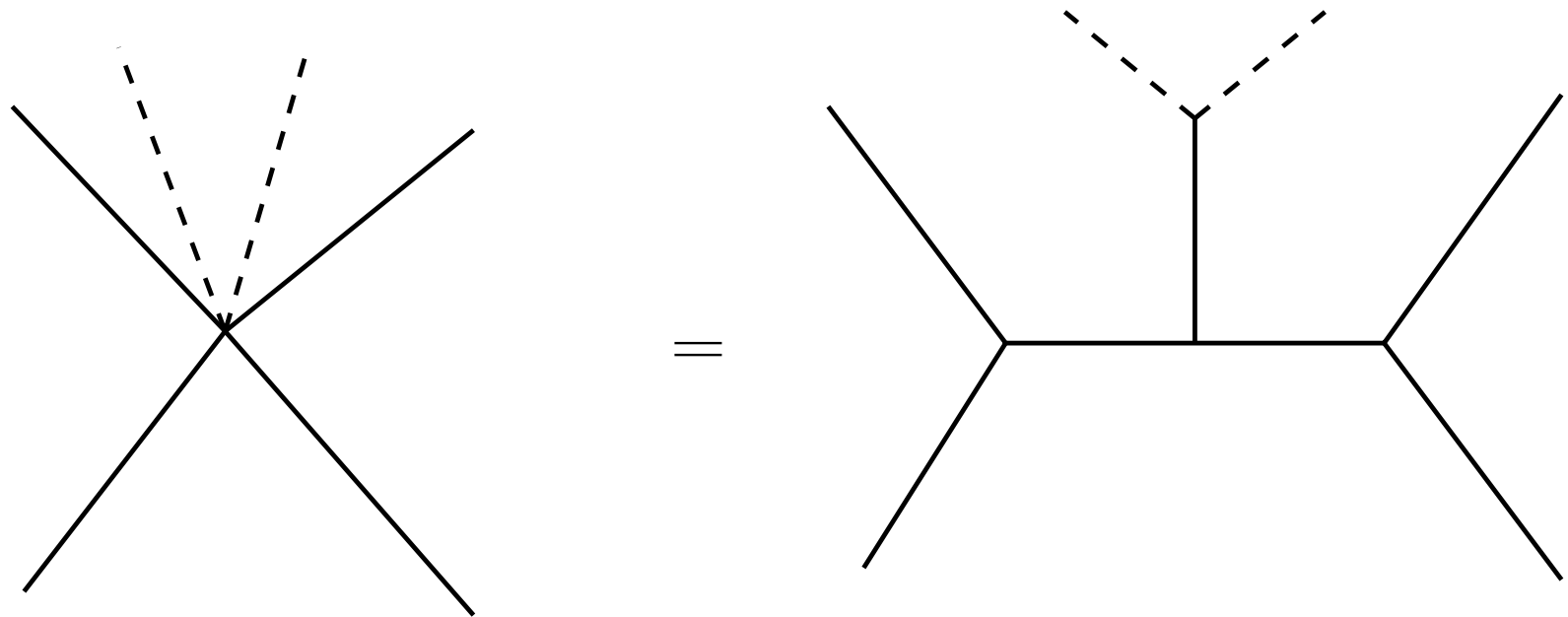


Figure VII: Diagrammatic representation of the equation (25).

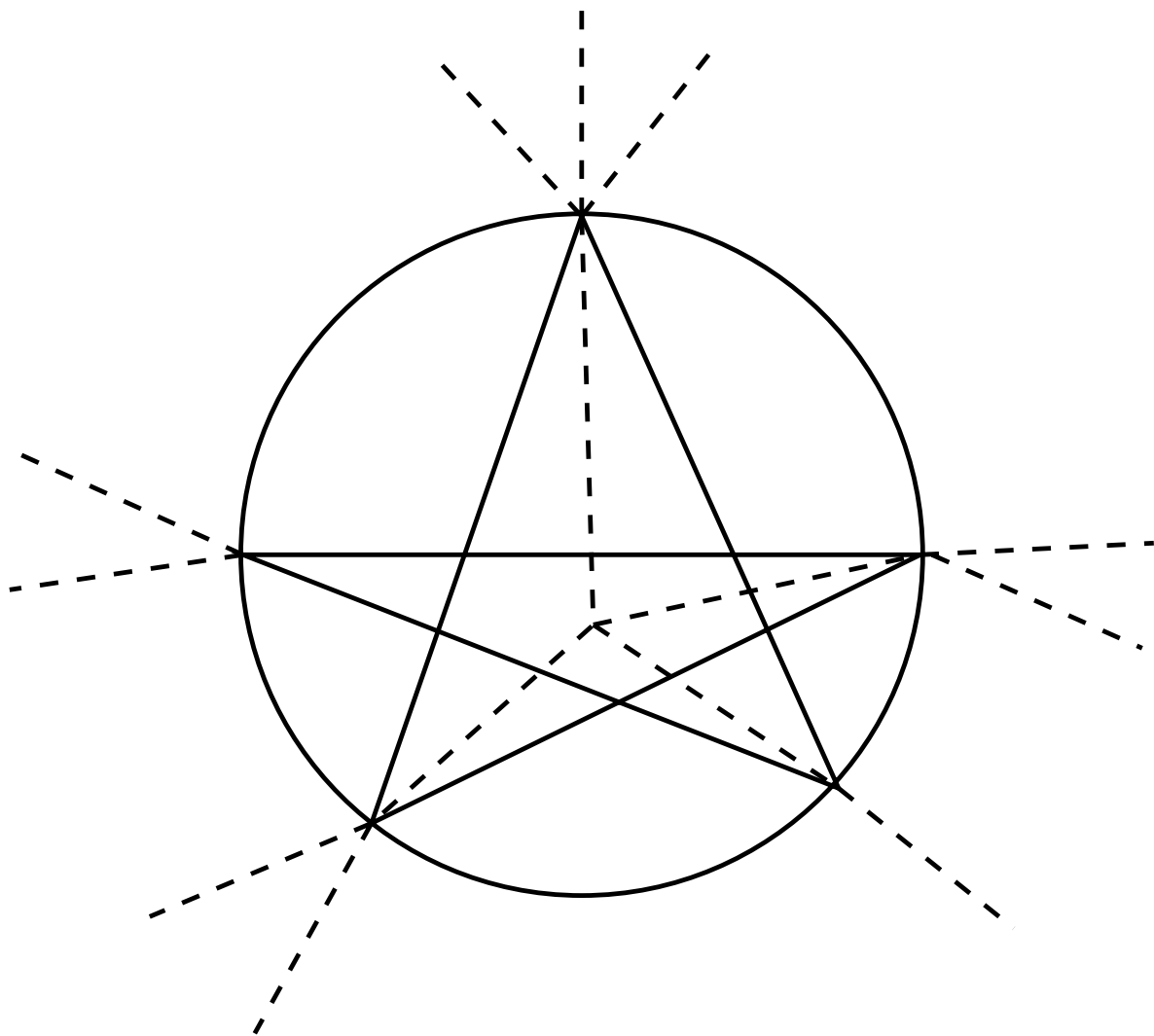


Figure VIII: A spin net graph determining a 15-valent spin foam vertex.

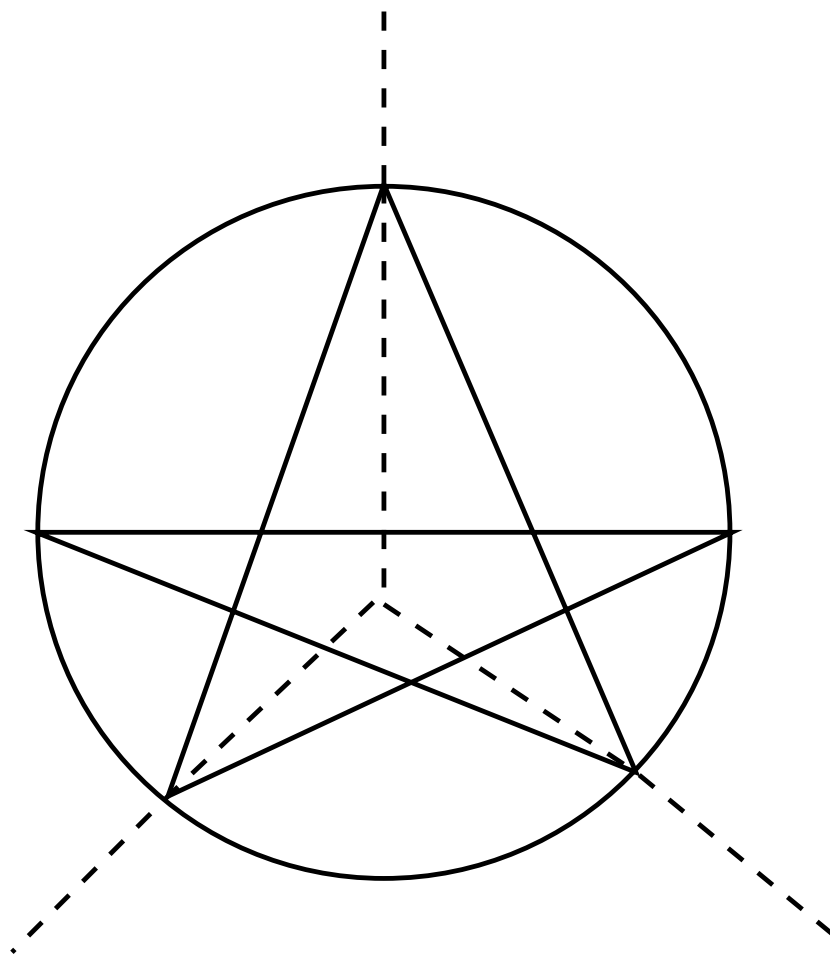


Figure IX: The spin net graph for the spin foam vector-spinor interaction vertex.